The nonlinear Poisson equation via a Newton-imbedding procedure

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Abstract

This article considers the semilinear boundary value problem given by the Poisson equation, $-\Delta u = f(u)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. For the zero boundary value case, we approximate a solution using the Newton-imbedding procedure. With the assumptions that f, f', and f'' are bounded functions on $\mathbb{R}$, with $f' < 0$, and $\Omega \subset \mathbb{R}^3$, the Newton-imbedding procedure yields a continuous solution. This study is in response to an independent work which applies the same procedure, but assuming that f' maps the Sobolev space $H^1(\Omega)$ to the space of Hölder continuous functions $C^\alpha(\bar{\Omega})$, and $f(u)$, $f'(u)$, and $f''(u)$ have uniform bounds. In the first part of this article, we prove that these assumptions force f to be a constant function. In the remainder of the article, we prove the existence, uniqueness, and H^2-regularity in the linear elliptic problem given by each iteration of Newton's method. We then use the regularity estimate to achieve convergence.

0 Introduction

The goal of this article is to find suitable hypotheses on a function $f \in C^2(\mathbb{R})$ related to attaining a solution to the semilinear boundary value problem given by

$$(*)\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u|_\Gamma &= \phi \quad \text{on } \Gamma = \partial\Omega, \end{cases}$$

using the Newton-imbedding procedure that is applied in [2]. Here, $f(u)$ is defined as $f \circ u$. *In this sense f can be viewed as a map from a space of real-valued functions to another space of real valued functions via composition.* In addition, $H^k(\Omega)$ is defined as the L^2 functions on Ω having (weak) i^{th} derivatives $(1 \leq |i| \leq k)$ which are L^2 functions on Ω. This is the Hilbert space notation substituted for the Sobolev space notation $W^{k,2}(\Omega)$. The space of real-valued functions on Ω which are Hölder continuous with exponent α will be denoted $C^\alpha(\bar{\Omega})$. The author of [2] achieves an H^2 solution when Ω is a domain in $\mathbb{R}^3$ and Γ is smooth, provided the following assumptions on f hold:

- 1. f is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$.
- 2. f' and f'' are continuous maps from $H^1(\Omega)$ to $C^\alpha(\bar{\Omega})$, $\alpha \in (0, \frac{1}{2}]$.

- 3. There exists a constant $M > 0$ such that

$$||f(u)||_{L^2(\Omega)} \leq M \text{ for all } u \in H^2(\Omega), \quad ||f'(u)||_{C^\alpha(\bar{\Omega})} \leq M \text{ for all } u \in H^1(\Omega),$$

$$\text{and } ||f''(u)||_{C^\alpha(\bar{\Omega})} \leq M \text{ for all } u \in H^1(\Omega).$$

- 4. $(-f')$ is positive in the sense that $(-f'(u)v, v) > 0$ for all $0 \neq v \in H^2(\Omega)$

An additional condition in [2] is the choice of a *uniform width* of time intervals in the procedure that ensures convergence, which exists as a consequence of the above assumptions. However, we prove the following theorems in Sections 2 and 3 of this article:

Theorem. 2.1 *If $f : \mathbb{R} \to \mathbb{R}$ is a map from $H^1(\Omega)$ to $C^0(\bar{\Omega})$ via composition and Ω is a domain in $\mathbb{R}^n$ with $n > 2$, then f is a constant function.*

Theorem. 3.1 *Let $h : \mathbb{R} \to \mathbb{R}$ map $H^2(\Omega) \bigcap H_0^1(\Omega)$ to $L^p(\Omega)$ via composition, where $1 \leq p \leq \infty$ and Ω is domain in R^n. If there exists a constant $M > 0$ such that $||h(u)||_{L^p} \leq M$ for all $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$, then h is a bounded function on $\mathbb{R}$, i.e. there exists a constant $C > 0$ such that $|h(x)| \leq C$ for all $x \in \mathbb{R}$.*

By Theorem 2.1, the assumption in (2) that f' maps H^1 to C^α forces f' to be a constant function. Theorem 3.1 shows that the uniform bound on $f(u)$ in assumption (3) forces f to be a bounded function on $\mathbb{R}$. Thus f is shown to be linear and bounded on $\mathbb{R}$, and is therefore a constant function, reducing the scope of the procedure in [2] to the family of problems given by $-\Delta u = const.$

In Section 1 of this article, we construct a '*mesa*' function (see Figure 1 in Section 1) whose existence in $H^1(\Omega)$ will serve as a counterexample to a non-constant mapping. In Section 2, the mesa function is used to prove Theorem 2.1. In Section 3, Theorem 3.1 is proven using a sequence of smooth 'bump' functions in H^2. As a consequence of this, the uniform bounds also imposed in (3) on $f'(u)$ and $f''(u)$ imply that f' and f'' are also bounded functions on $\mathbb{R}$. In Section 4, we describe and apply the Newton-imbedding procedure to the case of $(*)$ with a zero boundary condition. Of primary importance in the procedure is the following linear boundary value problem,

$$(**) \begin{cases} -\Delta u + q(x)u &= g(x) \quad \text{in } \Omega \\ u|_\Gamma &= 0 \quad \text{on } \Gamma, \end{cases}$$

given by each iteration in the Newton-imbedding procedure. Here, $q(x)$ is a positive scaling of $(-f')$ while $g(x)$ depends on f and f' in a manner that allows $g \in L^2$ under our assumptions. The exact hypotheses on q and g will be made precise in Section 4. As in [2], the assumption that $q > 0$ allows for existence and uniqueness for $(**)$ in H^1, as well as the regularity lifting of the H^1 solution to H^2. For the remainder of the article, it will be understood that (**) is the general boundary value problem stated above, with the conditions that $g \in L^2$ and $q > 0$. Under the following assumptions on f,

- I. f is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$.
- II. f' and f'' are continuous maps from $H^1(\Omega)$ to $L^n(\Omega)$
- III. there exists a constant $M > 0$ such that

$$|f| \le M, \quad |f'| \le M, \quad \text{and} \quad |f''| \le M.$$

- IV. $(-f') > 0$,

we prove existence and uniqueness for (**) in Section 5, and achieve the regularity lifting of an H_0^1 solution of (**) to H^2 in Section 6. These results are summarized in the following theorem:

Theorem. 6.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and $n > 2$. Then for $g \in L^2(\Omega)$, $q \in L^n(\Omega)$, and $q > 0$, the linear boundary value problem*

$$(**) \begin{cases} -\Delta u + q(x)u &= g(x) \qquad \text{in } \Omega \\ u|_\Gamma &= 0 \qquad \text{on } \Gamma, \end{cases}$$

has a unique solution $u \in H^2(\Omega) \bigcap H_0^1$ with

$$||u||_{H^2(\Omega)} \le C(||g||_{L^2(\Omega)}),$$

where C depends only on Ω, n, and q.

In Section 7, under an additional assumption (V) concerning the uniform width of time intervals in the procedure, convergence in the procedure is achieved resulting in the following theorem:

Theorem. 7.1 *With Ω a bounded domain in $\mathbb{R}^3$ with smooth boundary and assumptions (I)-(V), the semilinear boundary value problem,*

$$(*') \begin{cases} -\Delta u &= f(u) \qquad \text{in } \Omega \\ u|_\Gamma &= 0 \qquad \text{on } \Gamma = \partial\Omega, \end{cases}$$

has a unique solution in $H^2(\Omega) \bigcap H_0^1(\Omega)$, and hence a continuous solution, which can be approximated by the Newton-imbedding procedure.

1 The Mesa Function

Let Ω be a domain in $\mathbb{R}^n$ with $n > 2$ and let $c \in \Omega$. Since the function will be radially symmetric about c, define $r = \mid x - c \mid$ for $x \in \Omega$, and $T > 0$ such that $B(c, T) \subset\subset \Omega$, where $B(c, T)$ denotes the open ball of radius T about c. Also let $a, b \in \mathbb{R}$ with $a < b$, and $\alpha \in (0, \frac{n-2}{2})$. In order to define the function, it is

necessary to decompose the interval $[0, T]$ as follows:

If we let $r_1^+ = \dfrac{T}{2}$, then there is an s_1^+ such that

$$\frac{1}{(s_1^+)^\alpha} - \frac{1}{(r_1^+)^\alpha} = b - a.$$

In particular, $0 < s_1^+ < r_1^+$. Setting $s_1^- = \dfrac{s_1^+}{2}$ allows for an r_1^- such that

$$\frac{1}{(r_1^-)^\alpha} - \frac{1}{(s_1^-)^\alpha} = b - a.$$

In particular, $0 < r_1^- < s_1^-$. Continuing in this manner, set $r_{m+1}^+ = \dfrac{r_m^-}{2}$.

Note that $r_{m+1}^+ > 0$ for all m and r_{m+1}^+ goes to zero with $\dfrac{1}{2^m}$.

Using the above notation, let $U : \Omega \to$ R be the radially symmetric piecewise function defined inductively by

$$U(r) = \begin{cases} 0 & , \quad r \geq T \\ (\frac{-2a}{T})r + 2a & , \quad r_1^+ \leq r \leq T \\ \frac{1}{r^\alpha} - \frac{1}{(r_m^+)^\alpha} + a & , \quad s_m^+ \leq r \leq r_m^+ \\ b & , \quad s_m^- \leq r \leq s_m^+ \\ b - (\frac{1}{r^\alpha} - \frac{1}{(s_m^-)^\alpha}) & , \quad r_m^- \leq r \leq s_m^- \\ a & , \quad r_{m+1}^+ \leq r \leq r_m^-. \end{cases}$$

We will call $U(r)$ a *mesa* function with exponent α. Figure 1, below, is a sketch of a mesa function whose partition points have been altered to show more 'mesas'.

U is bounded and has compact support, so is trivially in $L^2(\Omega)$. It remains to show that it has (weak) first derivatives in $L^2(\Omega)$. The proposed first derivatives are given by

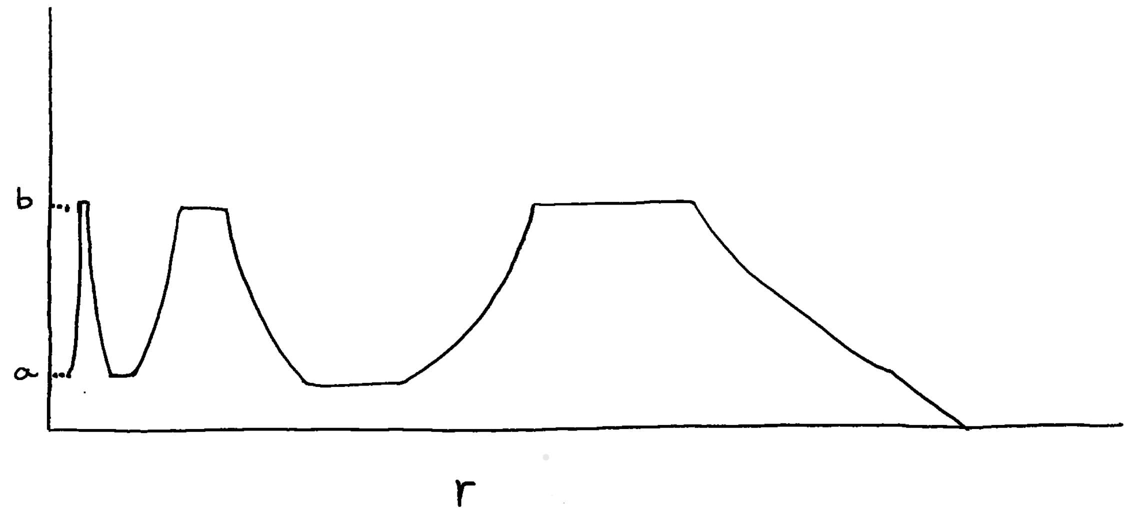

Figure 1: Artist's depiction of a mesa function

$$U_{x_i}(r) = \begin{cases} 0 & , \quad r \geq T \\ \frac{-2a}{T} & , \quad r_1^+ \leq r \leq T \\ \frac{-\alpha x_i}{r^{\alpha+2}} & , \quad s_m^+ \leq r \leq s_m^+ \\ 0 & , \quad s_m^- < r \leq s_m^+ \\ \frac{\alpha x_i}{r^{\alpha+2}} & , \quad r_m^- < r \leq s_m^- \\ 0 & , \quad r_{m+1}^+ < r \leq r_m^-. \end{cases}$$

Away from zero, on each annulus of the decomposed Ω, the expressions in U_{x_i} are classical derivatives of their corresponding expressions in $U(r)$. Let $\phi \in C_0^\infty(\Omega)$ and fix N. Integrating $U\phi_{x_i}$ by parts over the annuli given by $[r_1^+, T]$, $[s_m^+, r_m^+]$, $[s_m^-, s_m^+]$, $[r_m^-, s_m^-]$, and $[r_{m+1}^+, r_m^-]$ for $m = 1, ..., N$ and recalling that $U \equiv 0$ for $r \geq T$, gives

$$\int_{\Omega - B(c, r_{N+1}^+)} U\phi_{x_i} dx = -\int_{\Omega - B(c, r_{N+1}^+)} U_{x_i} \phi dx + \int_{\partial B(c, r_{N+1}^+)} U\phi\rho^i dS,$$

where $\rho = (\rho^1, ..., \rho^n)$ is the inward pointing normal on $\partial B(c, r_{N+1}^+)$.

Let $u(r) = \dfrac{1}{r^\alpha}$. Note that $|U_{x_i}| \leq |u_{x_i}|$, so that $|DU| \leq |Du|$.

Following the line of argument [1, p.246] given by L. Evans, since $\alpha < n-1$, $|Du| = \dfrac{\alpha}{r^{\alpha+1}} \in L^1(\Omega)$ and therefore $|DU| \in L^1(\Omega)$.

Letting $N \to \infty$ (and thus $r^+_{N+1} \to 0$),

$$\left| \int_{\partial B(c, r^+_{N+1})} U\phi\rho^i dS \right| \leq \| U\phi \|_\infty \int_{\partial B(c, r^+_{N+1})} \rho^i dS \ \leq \ M(r^+_{N+1})^{n-1} \to 0,$$

hence $$\int_\Omega U\phi_{x_i} dx = -\int_\Omega U_{x_i}\phi dx.$$

Therefore U_{x_i} is a (weak) derivative of U. Moreover, since $\alpha < \dfrac{n-2}{2}$, following the arguement in [1, p.246], $\mid Du \mid \in L^2(\Omega)$ and thus $|DU| \in L^2(\Omega)$ and $U(r) \in H^1(\Omega)$. The following lemma summarizes the above discussion:

Lemma 1.1 *If Ω is a domain in $\mathbb{R}^n$ with $n > 2$, and $U(r)$ is a mesa function with exponent $\alpha < \dfrac{n-2}{2}$, then $U(r) \in H^1(\Omega)$.*

2 Constant Mapping

Theorem. 2.1 *If $f : \mathbb{R} \to \mathbb{R}$ is a map from $H^1(\Omega)$ to $C^0(\bar{\Omega})$ via composition and Ω is a domain in $\mathbb{R}^n$ with $n > 2$, then f is a constant function.*

Proof. Suppose on the contrary, that f is not constant and assumes distinct values at a and b. Without loss of generality, assume that $a < b$. Let $c \in \Omega$ and T be such that $B(c,T) \subset\subset \Omega$. Since $n > 2$, there exists α such that $0 < \alpha < \frac{n-2}{2}$. Let $U(r)$ be the mesa function centered at c, with exponent α, support in $B(c,T)$, and prescribed maximum and minimum, b and a, respectively. By the above lemma, $U(r)$ is in $H^1(\Omega)$. Using the notation in the previous section for the domain of $U(r)$, it holds that for any $\delta > 0$ there exists an N such that $[s^-_N, s^+_N] \subset B(c,\delta)$ and $[r^+_{N+1}, r^-_N] \subset B(c,\delta)$. Note that $f \circ U \equiv f(b)$ on $[s^-_N, s^+_N]$ and $f \circ U \equiv f(a)$ on $[r^+_{N+1}, r^-_N]$. Since the measure of the above intervals is strictly positive, $f \circ U$ has no continuous representative. In other words, the oscillations of $f \circ U$ do not diminish in any neighborhood of c. This contradicts the hypothesis that f maps U to a continuous function.

□

Now, as an immediate application of Theorem 2.1, the assumption in (2) that f' maps H^1 into continuous functions forces f' to be constant.

3 Uniform Bounds

For this Section we assume Ω is a domain in $\mathbb{R}^n$.

Theorem. 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ map $H^2(\Omega) \bigcap H_0^1(\Omega)$ to $L^p(\Omega)$ where $1 \leq p \leq \infty$. If there exists a constant $M > 0$ such that $||f(u)||_{L^p} \leq M$ for all $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$, then f is a bounded function on $\mathbb{R}$, i.e. there exists a constant $C > 0$ such that $|f(x)| \leq C$ for all $x \in \mathbb{R}$.*

Proof. Let $p < \infty$. Suppose on the contrary, that f is not bounded. Then there exists a sequence, $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}$ such that $|f(x_k)| > k$. Let $y_0 \in \Omega$ and r such that $B = B(y_0, r) \subset\subset \Omega$. Set $B_{\frac{1}{2}} = B(y_0, \frac{r}{2})$. Choose a smooth function, γ, such that $\gamma \equiv 1$ on $B_{\frac{1}{2}}$, $\gamma \equiv 0$ on $\Omega - B$, and $0 \leq \gamma \leq 1$. Define the smooth function u_k on Ω by $u_k = x_k\gamma$. Then $u_k \in H^2(\Omega) \bigcap H_0^1(\Omega)$ for all k and

$$||f(u_k)||_{L^p(\Omega)} \geq ||f(u_k)||_{L^p(B_{\frac{1}{2}})} = ||f(x_k)||_{L^p(B_{\frac{1}{2}})} > k|B_{\frac{1}{2}}|^{\frac{1}{p}}.$$

Choosing k_0 large enough such that $k_0|B_{\frac{1}{2}}|^{\frac{1}{p}} > M$ gives a contradiction. If $p = \infty$, a similar computation holds, choosing $k_0 > M$.

$\square$

Remark: Since the C^α norm has the L^∞ norm as a summand, Theorem 5.1 with $p = \infty$ suffices to show that a uniform bound on $||f(u)||_{C^\alpha}$ implies f is bounded. Therefore the assumptions made in [2], imply that f, f', and f'' are bounded functions. Moreover, under the same assumptions, as shown in the previous Section, f is linear. In this case f is a constant, reducing the scope of the procedure to problems given by $-\Delta u = const.$

4 Newton-imbedding Procedure

The Newton-imbedding procedure we wish to apply to

$$(*')\begin{cases} -\Delta u &= f(u) \quad \text{in } \Omega \\ u|_\Gamma &= 0 \quad \text{on } \Gamma = \partial\Omega, \end{cases}$$

has two parts. It is well described in [2], but recalled here for clarity. The procedure first imbeds the problem in a one-parameter family of problems,

$$-\Delta u = tf(u) \qquad \text{in } \Omega$$

with $u = 0$ on Γ and parameter $t \in [0, 1]$. We set

$$F_t(u) = \Delta u + tf(u).$$

Solving $(*')$ is then a matter of solving $F_1(u) = 0$. Let $u(x, t)$ be the solution to $F_t(u) = 0$. Starting with $t_0 = 0$, the problem is solved with solution $u(x, 0)$ in Ω.

Observe that with boundary value zero imposed, $u(x,0)$ is uniquely determined as $u(x,0) \equiv 0$. To solve $F_{t_1}(u) = 0$, $u(x,0)$ is taken as an initial approximation and the standard Newton's method is applied. With convergence, the solution $u(x,t_1)$ to $F_{t_1}(u) = 0$ is achieved. The function $u(x,t_1)$ is then used as an initial approximation for $F_{t_2}(u) = 0$ and so on for increasing times t_j. Thus the solutions are pushed along with increasing times using Newton's method with the goal of reaching $t = 1$ in finitely many time shifts. Let $u_0(x,t_j) = u(x,t_{j-1})$, the initial approximation for $F_{t_j}(u) = 0$ and $u_m(x,t_j)$ be the m^{th} iteration of Newton's method at time t_j. In the following discussion, the argument of the u_m's will be suppressed. We will also temporarily use the symbol D for the Frechet derivative in contrast to its usual use as the gradient. Note that

$$DF_{t_j}(u_m)[w] = \Delta w + t_j Df(u_m)[w] \quad \text{and} \quad Df(u_m)[w] = f'(u_m)w$$

for $w \in H^2(\Omega)$ and that the $(m+1)^{th}$ iterate in the Newton approximation is given by

$$DF_{t_j}(u_m)[u_{m+1} - u_m] = -F_{t_j}(u_m).$$

In this case, the $(m+1)^{th}$ iteration at time t_j yields the following linear problem:

$$(**) \left\{ \begin{array}{rcll} -\Delta u_{m+1} + (-t_j f'(u_m))(u_{m+1}) & = & t_j(f(u_m) - f'(u_m)u_m) & \text{in } \Omega \\ u_{m+1}|_\Gamma & = & 0 & \text{on } \Gamma. \end{array} \right.$$

This is the problem

$$(**) \left\{ \begin{array}{rcll} -\Delta u + q(x)u & = & g(x) & \text{in } \Omega \\ u|_\Gamma & = & 0 & \text{on } \Gamma, \end{array} \right.$$

stated in the introduction with

$$q = -t_j f'(u_m), \quad g = t_j[f(u_m) + f'(u_m)u_m], \quad \text{and} \quad v = u_{m+1}.$$

Initially, a weak solution in H_0^1 is desired, so it makes sense that u be in H_0^1 and that f and f' should be defined on H_0^1. However, as will be shown in Section 6, an H_0^1 solution to (**) is also in H^2. In light of this, f and f' need only be defined on H^2. Note that if f maps H^2 to L^2 and f' maps H^2 to L^n, then g is in L^2 for all dimensions $n > 2$, via the Sobolev imbedding theorem. Indeed, since u is in H^1, u is again in $L^{\frac{2n}{n-2}}$ and the Hölder inequality gives

$$\int_\Omega [f'(u)u]^2 \le C||f'(u)||_{L^n}^2 ||u||_{L^{\frac{2n}{n-2}}}^2.$$

To fullfill the positivity condition on q in (**), we impose that $-f' > 0$. Now, at each time $t_j > 0$ and for all m, the m^{th} step in the iteration at time t_j is a model for (**).

For the remainder of the article, we assume Ω is a bounded domain in $\mathbb{R}^{n>2}$ with smooth boundary Γ and make the following assumptions (I)-(IV) on the nonlinear function f:

- I. f is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$.
- II. f' and f'' are continuous maps from $H^1(\Omega)$ to $L^n(\Omega)$
- III. there exists a constant $M > 0$ such that

$$|f| \leq M, \quad |f'| \leq M, \quad \text{ and } \quad |f''| \leq M.$$

- IV. $(-f') > 0$.

Remark: There is a redundancy and lack of 'sharpness' in assumptions (I) and (II), given (III). Indeed, if the functions f, f', and f'' are bounded, they naturally map to bounded functions on Ω, and hence to $L^\infty(\Omega)$ which is contained in $L^p(\Omega)$ for all $p \geq 1$ since Ω is bounded. The reason for stating L^2 explicitly is that it is a *familiar* assumption for framing weak solutions to linear elliptic problems. The bounds on the functions are not necessary to existence and uniqueness in (**), nor to the regularity lifting of the H_0^1 solution to H^2 . Moreover, the L^2 hypothesis on f and the L^n hypothesis on f' are sufficient for existence and uniqueness and the regularity lifting. For a more general treatment of elliptic equations with measurable coefficients, see [3].

5 Existence and Uniqueness

For this Section, we assume (I), (II), and (IV). To prove existance and uniqueness for (**) in $H_0^1(\Omega)$ (H^1 functions with zero on the boundary), the Riesz Representation theorem is sufficient. We seek a unique solution in $H_0^1(\Omega)$. The associated energy form for (**) is

$$B(u, v) = \int_\Omega DuDv + quv.$$

It is well defined on $H_0^1(\Omega)$. Indeed, since $n > 2$ and $u, v \in H_0^1(\Omega)$, then $u, v \in L^{\frac{2n}{n-2}}(\Omega)$ by the Sobolev imbedding theorem. Also since Ω is bounded, if $q \in L^n(\Omega)$, then $q \in L^{\frac{n}{2}}(\Omega)$. Note that

$$\frac{2}{n} + \frac{n-2}{2n} + \frac{n-2}{2n} = 1.$$

Therefore by Hölder's inequality, quv is integrable over Ω with

$$\int_\Omega |quv| \leq ||q||_{L^{\frac{n}{2}}} ||u||_{L^{\frac{2n}{n-2}}} ||v||_{L^{\frac{2n}{n-2}}}.$$

This inequality combined with the Sobolev inequality

$$|||u||_{L^{\frac{2n}{n-2}}} \leq C||u||_{H_0^1}$$

gives

$$|B(u,v)| \leq C||u||_{H_0^1}||v||_{H_0^1}$$

where $C > 0$ is dependent on Ω, n, and $||q||_{L^{\frac{n}{2}}}$ but not on u and v. By the Poincaré inequality and the positivity of q, we have

$$||u||_{H_0^1}^2 \leq C\int_\Omega |Du|^2 \leq C\int_\Omega |Du|^2 + qu^2 = CB(u,u)$$

where $C > 0$ is dependent on n and Ω but not on u. Since $f \in L^2(\Omega)$, it is a bounded linear functional on $H_0^1(\Omega)$ [1]. Since $B(u,v)$ is an inner product on H_0^1, the Riesz Representation theorem provides a unique $u^* \in H_0^1(\Omega)$ such that

$$B(u^*,v) = \int_\Omega fv \qquad \text{for all } v \in H_0^1.$$

In other words, u^* is the unique weak solution to (**) in H_0^1.

6 Regularity

With the same hypotheses as in the previous Section, we wish to lift the regularity of the unique solution to (**) from H_0^1 to H^2, with the estimate controlled by the L^2 norm of $g(x)$. Theorem 6.3.4 (Boundary H^2-regularity) in [1] gives the desired regularity lifting of a solution to (**) when $q \in L^\infty$. However, the L^∞ condition is only used in factoring out $||q||_{L^\infty}$ from the following integral to find, for $u, v \in H^1$ and $\epsilon > 0$ in Cauchy's inequality,

$$\int |quv| \leq ||q||_{L^\infty}\int |uv| \leq C\left(\frac{1}{2\epsilon}||u||_{L^2}^2 + \frac{\epsilon}{2}||v||_{L^2}^2\right).$$

The L^n hypothesis on q provides,

$$\int |quv| \leq \frac{1}{2\epsilon}||cu||_{L^2}^2 + \frac{\epsilon}{2}||v||_{L^2}^2 \leq \frac{1}{2\epsilon}\left(||q||_{L^n}^2||u||_{L^{\frac{2n}{n-2}}}^2\right) + \frac{\epsilon}{2}||v||_{L^2}^2$$

$$\leq C\left(\frac{\epsilon}{2}||u||_{H^1}^2 + \frac{\epsilon}{2}||v||_{L^2}^2\right) \leq C\left(\frac{\epsilon}{2}||Du||_{L^2}^2 + \frac{\epsilon}{2}||v||_{L^2}^2\right)$$

by Hölder's inequality, the Sobolev imbedding theorem and Poincare's inequality. By the above estimates, we have also

$$\int (cu)^2 \leq M||Du||_{L^2}^2.$$

Following the line of reasoning in [1], the result for $q \in L^n$ is a sufficient replacement for the estimate for L^∞ to get the regularity estimate,

$$||u||_{H^2(\Omega)} \leq C\left(||g||_{L^2(\Omega)} + ||u||_{H^1(\Omega)}\right)$$

where C depends only on Ω and n and q. Now, recalling the second energy estimate above,

$$||u||^2_{H^1(\Omega)}(\Omega) \leq CB(u,u) = C\int_\Omega gu \leq C\left(\frac{1}{2}||g||^2_{L^2(\Omega)} + \frac{1}{2}||u||^2_{L^2(\Omega)}\right).$$

since u is a weak solution to (**). The last inequality is given by Cauchy's inequality with $\epsilon = 1$. Also since u is a unique solution, the L^2 norm of u is controlled by the L^2 norm of g by Theorem 6.2.6 in [1]. Therefore,

$$||u||_{H^2(\Omega)} \leq C||g||_{L^2(\Omega)},$$

where C depends only on Ω, n, and more *significantly*, q.

To summarize the results in Sections 5 and 6, we have:

Theorem. 6.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and $n > 2$. Then for $g \in L^2(\Omega)$, $q \in L^n(\Omega)$, and $q > 0$, the linear boundary value problem*

$$(**)\begin{cases} -\Delta u + q(x)u &= g(x) \quad \text{in } \Omega \\ u|_\Gamma &= 0 \quad \text{on } \Gamma, \end{cases}$$

has a unique solution $u \in H^2(\Omega) \bigcap H^1_0$ with

$$||u||_{H^2(\Omega)} \leq C(||g||_{L^2(\Omega)}),$$

where C depends only on Ω, n, and q.

7 Convergence

In the previous two Sections, it was shown that (**) is uniquely solvable in H^1 and the solution is *a priori* in H^2 with estimate controlled by the forcing term g. Recalling that (**) represents an arbitrary iteration of Newton's method at time t_j, the linear equation solved by the difference, $u_{m+1} - u_m$ for $m > 1$, is given by

$$\begin{aligned} &-\Delta(u_{m+1} - u_m) + (-t_j f'(u_m))(u_{m+1} - u_m) \\ &= t_j(f(u_m) - f(u_{m-1}) - f'(u_{m-1})(u_m - u_{m-1})) \qquad \text{in } \Omega \\ &u_{m+1} - u_m = 0 \qquad \text{on } \Gamma. \end{aligned}$$

This is (**) with

$$v = u_{m+1} - u_m, \quad q = -t_j f'(u_m),$$

$$g = t_j(f(u_m) - f(u_{m-1}) - f'(u_{m-1})(u_m - u_{m-1})),$$

and a zero boundary condition. Indeed, using the same argument as at the end of Section 3, it is clear that $g \in L^2$. For $m = 0$, by the definition of u_0 at time t_j, the problem satisfied by $u_1 - u_0$ is

$$\begin{aligned} -\Delta(u_1 - u_0) + (-t_j f'(u_0))(u_1 - u_0) & \\ = (t_j - t_{j-1}) f(u_0) \qquad & \text{in } \Omega \\ u_1 - u_0 = 0 \qquad & \text{on } \Gamma, \end{aligned}$$

and is again a model for (**). To facilitate the convergence estimates to follow, it will be helpful to use Taylor's theorem to simplify g. Similar to the application of a mean value theorem used in [2], for $m > 1$, g can be written as

$$g = t_j (u_m - u_{m-1})^2 \int_{(0,1)} f''(\tau u_m + (1-\tau) u_{m-1})(1-\tau) d\tau.$$

Theorem 6.1 and the boundedness of f'' give the estimate,

$$||u_{m+1} - u_m||_{H^2} \leq C||t_j (u_m - u_{m-1})^2 \int_{(0,1)} f''(\tau u_m + (1-\tau) u_{m-1})(1-\tau) d\tau||_{L^2}$$

$$\leq \frac{Ct_j M}{2} ||(u_m - u_{m-1})^2||_{L^2}$$

$$\leq \frac{Ct_j M}{2} ||(u_m - u_{m-1})||_{L^4}^2.$$

Before progressing with the estimate, it is important to discuss the dependence on dimension. For dimensions $n = 3$ and $n = 4$, the L^4 norm is controlled by the H^1 norm, by the Sobolev imbedding theorem, which in turn is controlled by the H^2 norm. For dimensions $n = 5, 6, 7$,and 8, the L^4 norm is controlled by the H^2 norm, via the more general Sobolev inequality [1,p.270]. The subsequent calculations do not depend on which dimension $n \in (3, 4, 5, 6, 7, 8)$ is assumed. However, only in dimension $n = 3$ does the general Sobolev theorem assure that our H^2 solution is indeed continuous. For $n = 5, 6, 7$, and 8, the H^2 solution is respectively, L^{10}, L^6, $L^{\frac{14}{3}}$, and L^4. To continue with the convergence estimate, for $n \in (3, 4, 5, 6, 7, 8)$, we have

$$\frac{Ct_j M}{2} ||(u_m - u_{m-1})||_{L^4}^2 \leq \frac{Ct_j M C_s}{2} ||(u_m - u_{m-1})||_{H^2}^2$$

where C_s is the constant from the Sobolev theorem and only depends on Ω and n. Since in Theorem 6.1, C depends on $||f'(u_m(x, t_j))||_{L^n}$ and hence m and t_j, we invoke the boundedness of f'. Therefore $||f'(u_m(x, t_j))||_{L^n}$ is bounded by some constant $C > 0$, uniformly over m and t_j. Let $K = \frac{CMC_s}{2}$. Inductively,

$$||u_{m+1} - u_m||_{H^2} \leq (t_j K ||u_1 - u_0||_{H^2})^{2^m - 1} ||u_1 - u_0||_{H^2}$$

and therefore for $s \in \mathbb{N}$,

$$||u_{m+s} - u_m||_{H^2} \leq [a^{2^{m+s-1}-1} + ... + a^{2^m - 1}]||u_1 - u_0||_{H^2}$$

where $a = t_j K||u_1 - u_0||_{H^2}$. If t_j is chosen such that $a < 1$, then the *positive* expression in brackets above is bounded from above by the tail end of a convergent geometric series, and therefore goes to zero as $m \to \infty$. We have now shown that u_m is a Cauchy sequence in the Banach space $H^2(\Omega)$, and therefore converges to some $u^* \in H^2(\Omega)$. As stated in [2], due to the continuity of f and the boundedness of f', it is clear that u^* satisfies

$$(*') \begin{cases} -\Delta u &= t_j f(u) \quad \text{in } \Omega \\ u|_\Gamma &= 0 \quad \text{on } \Gamma \end{cases}$$

almost everywhere and that the uniqueness of the solution u^* follows from the uniquenss of the solution $u_m(x, t_j)$ to (**) for each m and t_j. One additional assumption is necessary for t_j to be chosen as above, as well as for progressing to $t = 1$ in finitely many applications of Newton's method. Assumption (V) will be a condition on the width of the time intervals $t_j - t_{j-1}$. To make this precise we look at the the problem satisfied by $u_1 - u_0$ at time t_j and apply Theorem 6.1 and the boundedness of f and f' to estimate,

$$||u_1 - u_0||_{H^2} \leq C||(t_j - t_{j-1})f(u_0)||_{L^2}$$

$$\leq C(t_j - t_{j-1})||f(u_0)||_{L^2} \leq MC(t_j - t_{j-1}).$$

If $A = MC$, then A depends on the bounds on f and f', the volume of Ω, and n, but not on t_j. In the following inequality,

$$Kt_j||u_1 - u_0||_{H^2} \leq KAt_j(t_j - t_{j-1}) < 1,$$

the condition for convergence was that the leftmost expression be < 1. Since $t_j \leq 1$ for all j, it suffices to make the assumption (V):

- V. For each $j \geq 1$, $\quad t_j - t_{j-1} < \frac{1}{KA}$

As KA only depends on Ω, $p = 2$, n, and M, (and in particular, not j), KA gives a uniform bound on the time intervals, and therefore $t = 1$ is attainable after finitely many applications of Newton's method. When Ω is a domain in $\mathbb{R}^3$, the H^2 solution is then continuous by the general Sobolev imbedding theorem. We now list assumptions (I)-(V) and state the main result.

- I. f is a continuous map from $H^2(\Omega)$ to $L^2(\Omega)$.
- II. f' and f'' are continuous maps from $H^1(\Omega)$ to $L^n(\Omega)$
- III. there exists a constant $M > 0$ such that

$$|f| \leq M, \quad |f'| \leq M, \quad \text{and} \quad |f''| \leq M.$$

- IV. $(-f') > 0$,
- V. For each $j \geq 1$, $\quad t_j - t_{j-1} < \frac{1}{KA}$

Theorem. 7.1 *With Ω a bounded domain in $\mathbb{R}^3$ with smooth boundary and assumptions (I)-(V), the semilinear boundary value problem,*

$$(*')\left\{\begin{array}{rcll} -\Delta u & = & f(u) & \text{in } \Omega \\ u|_\Gamma & = & 0 & \text{on } \Gamma = \partial\Omega, \end{array}\right.$$

has a unique solution in $H^2(\Omega) \bigcap H_0^1(\Omega)$, and hence a continuous solution, which can be approximated by the Newton-imbedding method.

8 Conclusion

The goal for improving this procedure is to weaken the assumptions on f and f'. In particular, to eliminate the boundedness or equivalently the uniform boundedness of $f(u)$ and $f'(u)$. To do this requires a function f such that $f(u_m(x,t))$ does not grow too fast in L^2 norm as t increases and such that $f'(u_m(x,t))$ does not grow too fast in L^n norm as m and t increase. If the boundedness of f is dropped from the assumptions, a linear function would be allowed, but assumption (IV) would force it to be decreasing. Since the spectrum of $-\Delta$ is positive, (*') is then solved uniquely with $u \equiv 0$ (which is achieved vacuously in the procedure). An example of a function satisfying (I)-(IV) is

$$f(x) = cot^{-1}(x)$$

whose derivatives are

$$f'(x) = \frac{-1}{1+x^2} \quad \text{and} \quad f''(x) = \frac{2x}{(1+x^2)^2}.$$

Similarly, if $\epsilon > 0$, $A > 0$, and $h, k \in \mathbb{R}$, then

$$Acot^{-1}\left(\frac{x-h}{\epsilon}\right) + k$$

represents a family of functions, each of which satify (I)-(IV). A subset of this family, given by

$$f_\epsilon(x) = \frac{1}{\pi}cot^{-1}(\frac{x}{\epsilon}) - 1,$$

is of interest since

$$f_\epsilon(x) \to -H \qquad \text{as} \qquad \epsilon \to 0$$

$$f'_\epsilon(x) = \frac{-\epsilon}{\epsilon^2 + x^2} \to -\delta \qquad \text{as} \qquad \epsilon \to 0,$$

where H is the Heaviside function and δ is the Dirac delta function and the arrows imply at least pointwise convergence and possibly a more refined limit. It is natural to ask whether the Newton-imbedding procedure can be carried out in a distributional setting with $f = -H$ and whether f_ϵ produces a meaningul approximation to the Heaviside function for small ϵ. More generally, if $\mathcal{P}$ is the class of functions which satisfy (I)-(IV), it is of interest as to which functions exist in a suitable closure of $\mathcal{P}$. In this case, 'suitable closure' can be taken to mean one whose functions allow for the application of the Newton-imbedding procedure in possibly a distributional or more general setting, and produce a solution which can be approximated by applying the procedure to a function in $\mathcal{P}$.

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